CORRECTION TO "HARMONIC REPRESENTATIVES FOR CUSPIDAL COHOMOLOGY CLASSES"

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1. INTRODUCTION

In this note we give a correction to the proof of the main result of [1]. The proof of Lemma 4.1 in [1] was incorrect in its use of Harnack inequality as pointed out by P. Buser and E. Makover. P. Buser suggested a workaround presented here. I am very grateful to Buser and Makover for alerting me to the problem and for discussions that led to its resolution.

We will follow the notation and conventions established in the first three sections of [1]. This note should be regarded as a replacement for Section 4 of that paper.

2. HARMONIC DIFFERENTIALS ON A PUNCTURED DISK AND LAURENT SERIES

There is a one-to-one correspondence between harmonic differentials on a coordinate neighborhood U on a Riemann surface and holomorphic functions of the complex local coordinate where the differential ω corresponds to the function $f(z)$ and the two are related by

$$\omega = \text{Re}(f(z)dz). \tag{1}$$

Moreover, writing $z = x + iy$ we have

$$\|\omega\|_U^2 = \int_U \omega \wedge *\omega = \int_U (a^2 + b^2)\, dxdy = \int_U |f|^2\, dxdy. \tag{2}$$

Let ω be a harmonic differential on an annulus $A_{\rho,1} = \{z = x + iy \mid \rho^2 < x^2 + y^2 < 1\}$ with period p on the cycle surrounding the origin and $f(z)$ be the corresponding holomorphic function. Suppose that

$$f(z) = \sum_{-\infty}^{\infty} a_n z^n$$

is the Laurent series expansion of $f(z)$ and that $z = \rho e^{i\theta}$.

Lemma 2.1. *The following formulae hold.*

$$p = -2\pi \,\text{Im}(a_1) \tag{3}$$

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$$\|f\|^2_{A_{\rho,1}} = \sum_{n=-\infty}^{\infty} \|a_n z^n\|^2_{A_{\rho,1}}$$

(4)
$$= 2\pi \sum_{n=2}^{\infty} |a_{-n}|^2 \frac{\rho^{2(1-n)} - 1}{2(n-1)}$$

$$+ 2\pi |a_{-1}|^2 \ln(1/\rho)$$

$$+ 2\pi \sum_{n=0}^{\infty} |a_n|^2 \frac{1 - \rho^{2(n+1)}}{2(n+1)}$$

In addition, if $t = \theta/2\pi$, the holomorphic function corresponding to the harmonic differential pdt is $p/2\pi iz$.

Proof. The proof is elementary. The Laurent series converges uniformly on compact subsets of the annulus so that integration can be performed term by term. (3) follows from the fact that the period of $z^n \, dz$ is zero unless $n = -1$. (4) is proved by a calculation using the fact that the powers z^n form an orthogonal system of functions. For the last comment observe that

$$dt = \frac{1}{2\pi} d\theta = \mathrm{Re}\left(\frac{1}{2\pi iz} dz\right).$$

$\square$

3. Proof of Theorem 2.1 of [1]

We parametrize all cusps of our Riemann surface X as in [1] and cut them off at $r = R$ for a large value of the parameter R to obtain the surface X_R. Since we allow our surface to have expanding ends (funnels), we consider also the surfaces $\overline{X}$ and $\overline{X}_R$ obtained by adding circles at infinity to funnels. Let ω_R be the unique harmonic differential on $\overline{X}_R$ satisfying the absolute boundary conditions along $\partial \overline{X}_R$ and with periods prescribed by the cohomology class c in the statement of Theorem 2.1 of [1]. By Hodge theory (cf. [2]) ω_R minimizes the L^2 norm in its cohomology class. To use this observation effectively we will need a closed form of degree one on $\overline{X}$ which is equal to a constant multiple of dt on every cusp. Let C be one of the cusps and let p be the period of the cohomology class c along a horocycle on the cusp. The form $\omega_R - pdt$ has period zero along horocycles and is therefore exact on $C_{[-\ln 2, R]}$ so that $\omega_R - pdt = d\phi$ for a smooth function ϕ on $C_{[-\ln 2, R]}$. Let $g(t)$ be a smooth nonnegative function such that $g|_{(-\infty, -1/2]} \equiv 1$ and $g|_{(0,\infty)} \equiv 0$. Consider the form $pdt + d(g\phi)$. This form is equal to ω_R near $r = -\ln 2$ and is equal to pdt for $r > 0$. Performing this construction on each cusp and extending the resulting form to be equal to ω_R on the complement of the union of cusps we obtain a form α with desired properties. We thus have

Lemma 3.1. *There exists a closed differential α on $\overline{X}$ with prescribed periods such that on every cusp C^i α is equal to $p_i dt$.*

From now on it will be convenient to denote by C the union of all cusps C^i. Similarly, for an interval I, C_I will stand for the union of subsets of all cusps where the r parameter belongs to the interval I. In addition, we will write pdt for the form on C whose restriction to the cusp C^i is $p_i dt$.

The following lemma will imply that the differentials $\{\omega_R\}_{R \geq 0}$ form a "normal family."

Lemma 3.2. *For every $a \geq 0$ there exists a constant $m(a)$ independent of R such that for all $R \geq a$*

$$\|\omega_R\|_{X_a} \leq m(a).$$

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Proof. We begin by observing that the form dt is harmonic and satisfies absolute boundary conditions on $C_{[r_1, r_2]}$ for every $r_1 < r_2$. Therefore, as above, pdt minimizes the norm it its cohomology class on $C_{[a,R]}$. It follows that

$$(5) \qquad \|pdt\|_{C_{[a,R]}} \leq \|\omega_R\|_{C_{[a,R]}}$$

On the other hand,

$$\|\omega_R\|_{X_R} \leq \|\alpha\|_{X_R}.$$

Therefore

$$(6) \quad \|\omega_R\|^2_{X_a} + \|\omega_R\|^2_{C_{[a,R]}} \leq \|\alpha_R\|^2_{X_a} + \|\alpha\|^2_{C_{[a,R]}} = \|\alpha_R\|^2_{X_a} + \|pdt\|^2_{C_{[a,R]}}$$

since $\alpha = pdt$ on cusps. The conclusion follows from (5) if we set $m(a) = \|\alpha_R\|_{X_a}$. $\qquad \square$

As explained in [1], the lemma implies that there exists a sequence R_j approaching infinity such the the forms ω_{R_j} converge uniformly with all derivatives on compact subsets of $\overline{X}$ to a form ω which is harmonic, has prescribed periods and satisfies the absolute boundary conditions along boundaries of funnels. To finish the proof we need to investigate the behavior of ω in cusps.

Lemma 3.3. *There exists a constant m_0 such that for every $0 < r < R$*

$$\|\omega_R\|^2_{C_{[0,r]}} \leq m_0 + \|pdt\|^2_{C_{[0,r]}}.$$

Proof. We proceed as in (6).

$$\|\omega_R\|^2_{X_0} + \|\omega_R\|^2_{C_{[0,r]}} + \|\omega_R\|^2_{C_{[r,R]}} \leq \|\alpha\|^2_{X_0} + \|\alpha\|^2_{C_{[0,r]}} + \|\alpha\|^2_{C_{[r,R]}}$$

We drop the first term on the left and, since $\alpha = pdt$ in cusps, rewrite the resulting inequality as

$$(7) \qquad \|\omega_R\|^2_{C_{[0,r]}} \leq \|\alpha\|^2_{X_0} + \|pdt\|^2_{C_{[0,r]}} + \|pdt\|^2_{C_{[r,R]}} - \|\omega_R\|^2_{C_{[r,R]}}$$

which proves the conclusion in view of (5). $\qquad \square$

We now prove that $\omega - pdt$ is square-integrable in cusps. Passing to the limit as R_j approaches infinity in (7) we see that

$$(8) \qquad \|\omega\|^2_{C_{[0,r]}} \leq m_0 + \|pdt\|^2_{C_{[0,r]}}.$$

To simplify the notation we continue the proof as if there was only one cusp. We represent the cusp $C_{[0,\infty)}$ as the punctured unit disk. Under this equivalence $C_{[0,r]}$ is mapped onto the annulus $A_{\rho,1}$ with $\rho = \rho(r)$ approaching zero as r tends to infinity. By Lemma 2.1

$$\|pdt\|^2_{C_{[0,r]}} = 2\pi |\operatorname{Im}(a_{-1})|^2 \ln(1/\rho).$$

By (8), the difference $\|f\|^2_{A_{\rho,1}} - 2\pi |\operatorname{Im}(a_{-1})|^2 \ln(1/\rho)$ is bounded when ρ approaches 0. In view of the formula (4), this is possible only if $a_{-n} = 0$ for $n \geq 2$ and $\operatorname{Re}(a_{-1}) = 0$. Thus there are no negative powers of z in the Laurent series expansion of the function corresponding to the differential $\omega - pdt$ so that this function has a removable singularity at the origin. In particular this function is square-integrable on the punctured disk and therefore $\omega - pdt$ is square-integrable on cusps.

The proof of existence of ω with required properties is complete. The uniqueness is proved as in Section 4 of [1].

References

[1] JÓZEF DODZIUK, JEFFREY MCGOWAN, & PETER PERRY. Harmonic representatives for cuspidal cohomology classes. In *Number theory, analysis and geometry*, 161–168. New York: Springer, 2012.

[2] D. B. RAY & I. M. SINGER. R-Torsion and the Laplacian on Riemannian Manifolds. *Advances in Math.* **7** (1971), 145–210.